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THE NUMBER OF FACTORIZATIONS OF NUMBERS LESS THAN x INTO FACTORS LESS THAN y

BY

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ABSTRACT. Let $K(x, y)$ be the number in the title. There is a function $f(r)$, concave and decreasing with $f(0) = 2$ and $f'(0) = 0$ such that if $r = \sqrt{\log x}/\log y$ then as $x \to \infty$ with r fixed,

$$K(x, y) = x\exp\left(f(r)\sqrt{\log x} + O(\log\log x)^2\right).$$

The proof uses a uniform version of Chernoff's theorem on large deviations from the sample mean of a sum of N independent random variables.

1. Introduction. In counting factorizations we make no distinction between $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$ and $3 \cdot 2 \cdot 2$, and list four factorizations of 12: 12, $6 \cdot 2$, $4 \cdot 3$ and $3 \cdot 2 \cdot 2$.

Let $F(n)$ denote the number of such factorizations of n. MacMahon observed in about 1920 that $\prod_{d=2}^{\infty}(1 - d^{-s})^{-1} = \sum_{n=1}^{\infty} F(n)n^{-s}$. Shortly after that Oppenheim considered the average and maximum values of $F(n)$ over the integers from 1 to x [4]. He found

$$\frac{1}{x}\sum_{n=1}^{x} F(n) \cong \exp\left(2\sqrt{\log x}\right)/2\sqrt{\pi}\,(\log x)^{3/4},$$

as did Szekeres and Turán somewhat later [5]. Their proofs were complex analytic, arising from MacMahon's formula.

Here we are interested in what happens when factorizations with any large term are excluded. Let $F_y(n)$ be the number of ways to write n as a product of factors d, $2 \leq d \leq y$, and let $K(x, y) = \sum_{n=1}^{x} F_y(n)$. How does $K(x, y)$ decrease as y shrinks from x toward 1?

This question is evocative of the classic work of de Bruijn on $\Psi(x, y)$, the number of positive integers $n \leq x$ with no prime divisor $> y$. There are structural as well as psychological similarities, as both Ψ and K satisfy a similar recursion. Yet the similarity does not run very deep. We show, for instance, that for any fixed $u > 0$, $\lim_{x\to\infty} K(x, x^{1/u})/K(x, x) = 1$. By contrast, $\lim_{x\to\infty}\Psi(x, x^{1/u})/\Psi(x, x) < 1$ for $u > 1$, and the limit ratio approaches zero as $u \to \infty$. It turns out that where for $\Psi(x, y)$, $u = \log x/\log y$ is the crucial parameter, for $K(x, y)$ things depend on $r = \sqrt{\log x}/\log y$.

Thus the situation is analogous to an economy where the rich hold most of the wealth. Namely, when $u = \log x/\log y$ is large but $r = \sqrt{\log x}/\log y$ is not, most of

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the numbers less than x have been ruled out. The few with no prime factor $> y$ which remain still contribute enough that $K(x, y)$ is 99^{+} % of the whole count $K(x, x)$.

Our *main result* is that $K(x, y)$ is roughly $x \exp(f(r)\sqrt{\log x})$, where $f(r)$ is a certain concave decreasing function of $r = \sqrt{\log x}/\log y$, with $f(0) = 2$ and $f'(0) = 0$.

The idea behind our factorization count estimate is that the count is related to the probability of a large deviation from the sample mean of the sum of some independent random variables.

We have what appears to be a new result about such deviations. Its statement and proof comprise §§5 and 6, which may be read independently. Given N independent random variables Y_i uniformly distributed in $[-\frac{1}{2}, \frac{1}{2}]$, and a parameter a, we show roughly that the asymptotic estimate given by Chernoff's theorem as $N \to \infty$ for $\mathrm{Prob}(|\sum_1^N Y_i| \geq N|a|)$ holds in this case *uniformly* for $N \geq 1$ and $-\frac{1}{2} < a < \frac{1}{2}$ as a lower bound. It seems likely that the restriction to such simple Y's is unnecessary but we need no greater generality.

We owe the reader a road-map to a first reading. §2 establishes some notation, touches again on the history of related topics and gives a simple estimate for $K(x, y)$ which is reasonably accurate when y is on the order of $\log x$. §3 can be read independently (and should be skipped initially); it contains a proof that $K(x, y) \sim K(x, x)$ even for y as small as $\exp(\sqrt{\log x} \log\log x)$. With such a small y, most $n \leq x$ have some prime factor greater than y and can make no contribution to $K(x, y)$.

The proof of the main result begins in §4 with the establishment of the connection between $K(x, y)$ and a question of probability. §§5, 6 and 7 should be skimmed for definitions and notation. The text of these sections is largely devoted to the painful but unavoidable task of working out the calculus of $f(r)$ and related functions.

In §8 we return to the main question, now replete with knowledge of $f(r)$ et al., and prove a lower bound for $K(x, y)$. In §9 we give a like upper bound for $K(x, y)$, with a slightly weaker error term.

Our results can probably be generalized in various ways. For instance let $G_y(n)$ denote the number of ways to write n as a product of divisors d, $-y \leq d \leq y$, with $d \neq -1$, 0, or 1, and let $K_{\pm}(x, y) = \sum_{n=-x}^{x} G_y(n)$. It is not hard to estimate $K_{\pm}(x, y)$ as something like $x \exp(\sqrt{2} f(r/\sqrt{2})\sqrt{\log x})$. Similar things should hold for e.g. the Gaussian integers, where we expect that with

$$K_{\mathbf{C}}(x, y) = \#\left\{\xi\colon S_y\{a + bi\colon 1 < a^2 + b^2 \leq y\} \to \{0, 1, 2, \ldots\}\right.$$
$$\left.\text{such that } \sum_{z \in S_y} \xi_z \log|z| \leq \frac{1}{2}\log x\right\}$$

$K_{\mathbf{C}}(x, y)$ will be something like $x \exp(\sqrt{\pi} f(r/\sqrt{\pi})\sqrt{\log x})$.

2. The problem and some preliminary observations. A *factorization* of a natural number n is a solution in nonnegative integer exponents ξ_d of the equation

$$\prod_{d \geq 2} d^{\xi_d} = n.$$

Thus 1 has one factorization, and 12 has four.

Let $K(x, y)$ be the number of factorizations of integers $1 \leqslant n \leqslant x$ into factors $2 \leqslant d \leqslant y$. Equivalently,

$$K(x, y) = \#\left\{\xi\colon \{2,3,\dots,y\} \to \{0,1,2,\dots\} \text{ such that } \prod_2^y d^{\xi_d} \leqslant x\right\}.$$

We abbreviate $K(x, x)$ as $K(x)$. Later on we have to keep very small divisors apart, and we define

$$K(x, y, z) = \#\left\{\xi\colon \{z+1, z+2,\dots,y\} \to \{0,1,2,\dots\} \text{ such that } \prod_{z+1}^{y} d^{\xi_d} \leqslant x\right\}.$$

How does $K(x, y)$ decrease as y shrinks from x toward 1? We know about the endpoints. Oppenheim (1926), and later Turán and Szekeres, found that

$$K(x) \sim x\exp(2\log^{1/2} x)/\left(2\sqrt{\pi}\log^{3/4} x\right). \tag{2.1}$$

(Here and later, $\log_n^c x =$ the cth power of the nth logarithm of x.) Trivially $K(x, 1) = 1$, and for $y \leqslant \log x/\log_2^2 x$ the lower bound

$$K(x, y) \geqslant \binom{y - 1 + [\log x/\log y]}{y-1}$$

is fairly tight. (This lower bound is a direct adaptation of de Bruijn's for $\Psi(x, y)$, with the same proof. See [2].)

We can improve on this a bit with our first result. For $y \leqslant \log x/\log_2^2 x$ it is sharper by a factor on the order of $\exp(y/2\log y)$.

THEOREM 2.1. *For $x \geqslant y \geqslant 2$,*

$$K(x, y) \geqslant \left(\log^{y-1} x/(y-1)!\right)\prod_2^y (1/\log d).$$

PROOF. Every ξ counted in $K(x, y)$ is an integer lattice point of $\mathbf{R}^{y-1}$ and an element of the simplex $\Sigma_2^y x_d \log d \leqslant \log x$, which has volume given by the right side of Theorem 2.1. The union, taken over all ξ counted in $K(x, y)$, of the unit cubes $\xi_d \leqslant x_d \leqslant \xi_d + 1$ $(2 \leqslant d \leqslant y)$ contains the simplex.

By taking a bigger simplex that contains all these cubes we get an upper bound for $K(x, y)$ which is $(1 + \Sigma_2^y \log d/\log x)^{y-1}$ times the lower bound of Theorem 2.1.

3. For large y, $K(x,y) \sim K(x)$. We start with a simple recursion. For proof of a similar recursion see [2].

$$K(x, y_1) = K(x, y_2) - \sum_{y_1 < d \leqslant y_2} K(x/d, d), \tag{3.1}$$

for $1 \leqslant y_1 < y_2 \leqslant x$. Since $K(x, y) \leqslant K(x)$,

$$K(x, y) \geqslant K(x) - \sum_{y < d \leqslant x} K(x/d). \tag{3.2}$$

From (2.1), there exist $C_1 > 0$, $C_2 > 0$ such that $K(t) \leqslant C_1 t \exp(2\log^{1/2} t)\log^{-3/4} t$ for $t \geqslant e$ and so that the reverse inequality holds with C_2 in place of C_1. Thus

$$(3.3)\qquad \sum_{y<d\leqslant x} K(x/d) \leqslant \sum_{y<d\leqslant x/e} K(x/d) + 2x(1-1/e)$$
$$\leqslant C_1 x\int_y^{x/e} t^{-1}\exp\left(2\log^{1/2}(x/t)\right)\log^{-3/4}(x/t)\,dt + 2x.$$

Let $I(x, y)$ denote the integral in (3.3). Then

$$I(x,y) \leqslant \log^{1/4} x\int_{\log y/\log x}^{1-1/\log x}(1-s)^{-3/4}\exp\left(2((1-s)\log x)^{1/2}\right)ds.$$

Let $u = \log x/\log y$, $v = \log y\log_2^{-1} x\log^{-1/2} x$, and $J = \log^{1/2} x(1-1/u)^{-1/2}$. Then

$$2(1-s)^{1/2}\log^{1/2} x \leqslant 2(1-1/u)^{1/2}\log^{1/2} x - J(s-1/u),$$

so that

$$I(x,y) \leqslant C_3(1-1/u)^{-3/4}\exp\left(2(1-1/u)^{1/2}\log^{1/2} x\right)$$
$$\cdot\exp\left(v\log_2 x(1-1/u)^{-1/2}\right)\log^{1/4} x\int_{1/u}^{1-1/\log x} e^{-Js}\,ds$$
$$\leqslant C_3\log^{-1/4} x\exp\left(2\log^{1/2} x - v\log_2 x\right).$$

So

$$\sum_{y<d\leqslant x} K(x/d) \leqslant 2x + C_4 x\log^{-v-1/4} x\exp\left(2\log^{1/2} x\right)$$
$$\leqslant C_5 x\log^{-v-1/4} x\exp\left(2\log^{1/2} x\right).$$

Together with (2.1) this gives $\sum_{y<d\leqslant x} K(x/d) \leqslant C_6\log^{-v+1/2} xK(x)$, and with (3.2), we have

THEOREM 3.1. *For* $x \geqslant y$ *and* $v = \log y\log^{-1/2} x\log_2^{-1} x > \frac{1}{2}$, $K(x,y) \geqslant K(x)(1 - C_6\log^{-v+1/2} x)$. □

REMARK. In particular, $\lim_{x\to\infty} K(x, x^{1/u})/K(x) = 1$ for fixed u.

4. The probability connection. We now consider the case that $\log y = r^{-1}\log^{1/2} x$, where $r > 0$ is not "too large" compared to x. Our main theorem is that $K(x,y) = x\exp(f(r)\log^{1/2} x + O(\log\log x)^2)$ under suitable restrictions on r. Here $f(r)$ is a certain function f: $[0,\infty) \to (-\infty, 2]$ with $f(0) = 2$, $f'(0) = 0$ and f decreasing and concave down on $[0,\infty)$. For details on f see §7. For precise lower and upper bounds on $K(x, y)$ see §§8 and 9.

In this section we show how counting $K(x, y)$ is related to probability. Our approach is to group factorizations ξ of numbers near x according to how many of the factors lie in each of the intervals $[\alpha^{i-1}, \alpha^i]$ up to y. Then we sum over all possible interval counts.

Let α be that number nearest $\exp(\log^{-4} x)$ such that $B = \log y/\log\alpha$ is an integer, and let z be that number nearest $\log^6 x$ such that $A = \log z/\log\alpha$ is an integer. Then $\alpha = \exp(\log^{-4} x + O(\log^{-9/2} x))$, and $z = \log^6 x + O(\log^2 x)$. Very small factors turn out not to make much difference, so we start on a lower bound with $K(x, y) \geqslant K(x, y, z)$.

Let $W_i = (\alpha^{i-1}, \alpha^i]$, $A \leqslant i \leqslant B$, and let w_i be the number of integers in W_i. Thus $w_i = \alpha^i(1 - 1/\alpha) + O(1)$. Let $Y = \{z + 1, z + 2, \ldots, y\}$ and $N = \{0, 1, 2, \ldots\}$. For ξ: $Y \to N$ let $v_i = \sum_{d \in W_i} \xi_d$, and let $V_\xi = (v_{A+1}, v_{A+2}, \ldots, v_B)$. If $\sum_{A+1}^B i v_i \leqslant \log x / \log \alpha$ then $\sum_{d \in Y} \xi_d \log d \leqslant \log x$ and ξ is counted in $K(x, y, z)$. Thus

$$K(x, y, z) \geqslant \# \left\{ \xi \colon Y \to N \text{ such that } \sum_{A+1}^{B} i v_i \leqslant \log x / \log \alpha \right\}. \tag{4.1}$$

The number of such ξ with the same V_ξ is $\prod_{A+1}^B \binom{w_i + v_i - 1}{v_i}$, since the number of ways to put v balls in w urns is $\binom{w+v-1}{v}$. Now for any w and v,

$$(w^v/v!) \leqslant \binom{w + v - 1}{v} \leqslant (w^v/v!) \exp(v^2/w),$$

so the number of ξ with the same V_ξ is $\geqslant \prod_{A+1}^B (w_i^{v_i}/v_i!)$.

Now $w_i = (1 - 1/\alpha)\alpha^i + O(1)$, so $(w_i(1 - 1/\alpha)^{-1}\alpha^{-i}) = 1 + O(\log^{-2} x)$, and since $\prod_{A+1}^B v_i \leqslant \log x / \log z \leqslant \log x / \log_2 x$,

$$\prod_{A+1}^{B} w_i^{v_i}/v_i! \geqslant \frac{1}{2}(1 - 1/\alpha)^E \prod_{A+1}^{B} \alpha^{i v_i}/v_i!, \tag{4.2}$$

where $E = \sum_{A+1}^B v_i$.

Most factorizations in $K(x, y)$ have $\prod_2^y d^{\xi_d}$ near x, so we should be able to replace $\prod_{A+1}^B \alpha^{i v_i}$ with x and not increase the right side of (4.2) by much. We can, but it is a long story, which we defer to §8, in part. If we neglect this difficulty temporarily, the other thing in (4.2) we must understand is $\prod_{A+1}^B 1/v_i!$.

Consider the set S_E of all $V = [v_{A+1}, \ldots, v_B]$ such that $\sum_{A+1}^B v_i = E$. If $E > u$ $(= \log x / \log y)$ we cannot have $\prod \alpha^{i v_i} \leqslant x$ for all $V \in S_E$.

Let $S_E(t) = \{V \in S_E\colon \sum_{A+1}^B i v_i \leqslant t\}$. Let $Q_E(t) = \sum_{S_E(t)} \prod_{A+1}^B (1/v_i!)$, and let $P_E(t) = Q_E(t)/Q_E(\infty)$. Then $0 \leqslant P_E(t) \leqslant 1$, and if $X_1, X_2, \ldots, X_E$ are independent integer valued random variables with probability $1/(B - A)$ on $\{A + 1, A + 2, \ldots, B\}$ then we have for any $E \geqslant 1$, any A and B with $A < B$ and any t,

THEOREM 4.1. $\operatorname{Prob}(\sum_1^E X_n \leqslant t) = P_E(t)$.

PROOF. First we show

$$Q_E(\infty) = (B - A)^E / E!. \tag{4.3}$$

For if $B - A = J = 1$, then $Q_E(\infty) = 1/E! = (J^E/E!)$. If (4.3) holds for $j \leqslant J$ and if $B - A = J + 1$ then

$$Q_E(\infty) = \frac{1}{E!} + \frac{1}{(E-1)!}\frac{J^1}{1!} + \cdots + \frac{1}{1!}\frac{J^E}{E!}$$

$$= \frac{1}{E!} \sum_{n=0}^{J} \binom{E}{n} J^n = \frac{1}{E!}(J + 1)^E. \quad \square$$

Now for $V \in S_E$ let $E(V)$ be the event that exactly v_k of the X_n's are equal to k, for each k, $A+1 \leqslant k \leqslant B$. The number of primitive events in $E(V)$ is

$$\prod_{A+1}^{B} \binom{E - \sum_{A+1}^{n-1} v_i}{v_n} = E! \prod_{A+1}^{B} (1/v_n!),$$

so

$$\operatorname{Prob}(E(V)) = (B-A)^{-E} E! \prod_{A+1}^{B} 1/v_n!,$$

and

$$\sum_{S_E(t)} \prod_{A+1}^{B} 1/v_n! = (B-A)^E (E!)^{-1} \operatorname{Prob}\left(\sum_{1}^{E} X_n \leqslant t \right).$$

COROLLARY 4.2. *Let* $S_E(t_1, t_2) = \{V \in S_E \colon \Sigma_{A+1}^{B} i v_i \in (t_1, t_2]\}$, *and let* $P_E(t_1, t_2) = P_E(t_2) - P_E(t_1)$. *Then* $P_E(t_1, t_2) = \operatorname{Prob}(t_1 < \Sigma_1^E X_n \leqslant t_2)$ *and* $\Sigma_{S_E(t_1, t_2)} \Pi_{A+1}^{B} (1/v_n!) = (B-A)^E (E!)^{-1} P_E(t_1, t_2)$.

REMARK. In the application, $t_2 = [\log x / \log \alpha]$ and $t_1 = t_2 - [\log_2 x / \log \alpha] + 1$. Also $E = [h(r) \log^{1/2} x]$ is chosen to maximize the bound for $K(x, y)$ that arises from Corollary 4.2. For a description of $h(r)$ see §7. With these values of t_1 and t_2,

$$\begin{aligned} K(x, y) &\geqslant K(x, y, z) \\ &\geqslant \sum_{E} \sum_{V \in S_E(t_1, t_2)} \frac{1}{2} (1 - 1/\alpha)^E \prod_{A+1}^{B} \alpha^{i v_i} / v_i! \\ &\geqslant \frac{1}{2} x \log^{-1} x (1 - 1/\alpha)^E \sum_{S_E(t_1, t_2)} \prod_{A+1}^{B} 1/v_i! \quad \text{(for any particular } E\text{)} \\ &= \frac{1}{2} x \log^{-1} x (1 - 1/\alpha)^E (B-A)^E (E!)^{-1} P_E(t_1, t_2) \\ &\geqslant \frac{1}{4} x \log^{-1} x (\log y - \log z)^E (E!)^{-1} P_E(t_1, t_2) \end{aligned}$$

and since $E \leqslant \log x$,

$$K(x, y) \geqslant \frac{1}{4} x \log^{-3} x (\log y - \log z)^E E^{-E} e^{+E} P_E(t_1, t_2). \tag{4.4}$$

We now get a lower bound for P_E and take E to more or less maximize the right side of (4.4).

5. Calculus of large deviations. To do the necessary calculations, we shall need to introduce a chain of probabilistic variables. In order of appearance, these are a, τ (5.1), ρ (5.6), σ (5.9), β (5.10), and finally the promised function $f(r)$ in §7. The uniform Chernoff's theorem, given in §6, is stated in terms of the function $\rho(a)$ of the parameter a, which also appears in [**1**]. Intuitively, the significance of ρ is that in N independent trials of a random draw from $[-\frac{1}{2}, \frac{1}{2}]$ the sum is unlikely to fall outside $(-Na/2, Na/2)$. The odds of this drop exponentially in N, like $(\rho(a))^N$.

Let W_n, $1 \leqslant n \leqslant N$, be independent, identically distributed random variables each with density $\lambda(s) = 1$ for $-\frac{1}{2} \leqslant s \leqslant \frac{1}{2}$ and 0 otherwise. For $-\frac{1}{2} < a < \frac{1}{2}$ and $N \geqslant 1$ let $P(a, N) = \text{Prob}(|\sum_1^N X_i| > N|a|)$. By a theorem of Bernstein [**1**], $P(a, N) \leqslant (\rho(a))^N$, where ρ is the function given by (5.6) below, and in [**1**]. In the other direction, from Chernoff's theorem [**1**] for *fixed a*, $\lim_{N\to\infty} \frac{1}{N} \log P(a, N) = \log \rho(a)$. We make this uniform in a, $-\frac{1}{2} < a < \frac{1}{2}$. We defer an exact statement of the theorem and its proof to §6 as we shall need a mass of information about ρ and related quantities first.

We start by defining τ: $(-\frac{1}{2}, \frac{1}{2}) \to (-\infty, \infty)$. Let $\tau(a)$ be the unique real number τ such that $\int_{-1/2-a}^{1/2-a} xe^{\tau x}\, dx = 0$. Thus $\tau(0) = 0$, $\tau(a) > 0$ for $0 < a < \frac{1}{2}$, and τ is continuous on $(-\frac{1}{2}, \frac{1}{2})$. Also τ is odd.

(5.1) For $a \neq 0$, if $\tau = \tau(a)$ then

$$a = \frac{1}{2}\frac{\cosh(\tau/2)}{\sinh(\tau/2)} - \frac{1}{\tau}.$$

PROOF. With this value for a, the defining integral works out to zero.

REMARK. Using (5.1) to express τ in terms of a, $\cosh(\tau/2)$ and $\sinh(\tau/2)$ gives a recursion which tends to $\tau(a)$. Newton's method requires a longer program but converges faster.

(5.2) On $(-\frac{1}{2}, \frac{1}{2})$, τ is increasing, and $\lim_{a\to 1/2} \tau(a) = +\infty$.

PROOF. If $-\frac{1}{2} < a < b < \frac{1}{2}$ and $\int_{-1/2-a}^{1/2-a} xe^{x\tau(a)}\, dx = 0$ then $\int_{-1/2-b}^{1/2-b} xe^{x\tau(a)}\, dx < 0$. Since $\int_{-1/2-b}^{1/2-b} xe^{x\tau(b)}\, dx = 0$, $\int_{-1/2-b}^{1/2-b} x(e^{x\tau(b)} - e^{x\tau(a)})\, dx > 0$ so $\tau(b) > \tau(a)$. For the second claim, note that for fixed τ,

$$\lim_{a\to 1/2} \int_{-1/2-a}^{1/2-a} xe^{\tau x}\, dx = \int_{-1}^{0} xe^{\tau x}\, dx < 0. \quad \square$$

Let a: $(-\infty, \infty) \to (-\frac{1}{2}, \frac{1}{2})$ be the inverse of τ. Then $a(0) = 0$ and for $\tau \neq 0$,

$$a(\tau) = \frac{1}{2}\frac{\cosh(\tau/2)}{\sinh(\tau/2)} - \frac{1}{\tau}.$$

(5.3) For $-\frac{1}{2} < a < \frac{1}{2}$ and $a \neq 0$, $da/d\tau = \tau^{-2} - (4\ \sinh^2(\tau/2))^{-1} > 0$. At 0, $da/d\tau = \frac{1}{12}$.

PROOF. The derivative is calculated from the formula given above for $a(\tau)$. Since $|\sinh(\tau/2)| > |\tau/2|$ it is positive. From L'Hôpital's rule, $da(0)/d\tau = \lim_{\tau\to 0}(a/\tau) = \lim_{\tau\to 0}(da/d\tau)$. Near 0, $da/d\tau = \tau^{-2}(1 - (1 + \tau^2/12 + O(\tau^4))^{-1}) = \frac{1}{12} + O(\tau^2)$.

Thus $\lim_{\tau\to 0} a/\tau = \frac{1}{12}$ and $\lim_{a\to 0} \tau/a = 12$.

(5.4) For $\tau \neq 0$, $d^2a/d\tau^2 = -2\tau^{-3} + \frac{1}{4}\cosh(\tau/2)\sinh^{-3}(\tau/2)$. On $(0, \infty)$, $d^2a/d\tau^2 < 0$ and $a(\tau)$ is concave, on $(-\infty, 0)$, $d^2a/d\tau^2 > 0$ and $a(\tau)$ is convex, and $(d^2a/d\tau^2)(0) = 0$.

PROOF. For $\tau \neq 0$, $d^2a/d\tau^2$ is calculated from (5.3). Since τ is odd so is $d^2a/d\tau^2$. That $d^2a/d\tau^2 < 0$ on $(0, \infty)$ follows from $8\sinh^3(\tau/2) > \tau^3\cosh(\tau/2)$ for $\tau > 0$, or equivalently $\sinh^3(s) > s^3\cosh(s)$ for $s > 0$. We prove this by comparing power series.

We divide by s^3 and need $(1 + s^2/3! + s^4/5! + \cdots)^3 > (1 + s^2/2! + s^4/4! + \cdots)$. It is sufficient to show that the s^{2n} coefficient on the left exceeds that on the right for $n \geq 1$. There are three cases: $n \equiv 0$, 1 or 2 mod 3.

If $2n = 6m$, say, then the s^{2n} term on the left includes $(s^{2m})^3((2m+1)!)^{-3} > s^{6m}/(6m)!$ since by induction on m, $((2m+1)!)^3 < (6m)!$ for $m \geq 1$. The other cases are left to the reader.

Now at 0, $d^2a/d\tau^2 = \lim_{\tau\to 0} \tau^{-1}(da/d\tau - \frac{1}{12})$ and $da/d\tau = \frac{1}{12} + O(\tau^2)$ from the proof of (5.3) so this limit is zero. □

(5.5) $-1 < (\frac{1}{2} - |a|)\tau(a) < 1$ and $\lim_{a\to 1/2}(\frac{1}{2} - a)\tau(a) = 1$.

PROOF. Since τ is odd it is sufficient to prove this for $0 < a < \frac{1}{2}$. Now

$$\left(\frac{1}{2} - a\right)\tau(a) = 1 - \frac{1}{2}\tau\left(\frac{\cosh(\tau/2)}{\sinh(\tau/2)} - 1\right) < 1$$

and $\cosh(\tau/2)/\sinh(\tau/2) = 1 + O(e^{-\tau})$ as $\tau \to \infty$ when $a \to \frac{1}{2}$, so

$$\frac{1}{2}\tau\left(\frac{\cosh(\tau/2)}{\sinh(\tau/2)} - 1\right) = O(\tau e^{-\tau}) \to 0$$

as $a \to \frac{1}{2}$ and $\tau \to \infty$. □

The $\rho(a)$ of the introduction to §5 can now be defined.

DEFINITION. For $-\frac{1}{2} \leq a \leq \frac{1}{2}$ let $\rho = \rho(a) = \inf_t \int_{-1/2-a}^{1/2-a} e^{tx}\,dx$.

(5.6) For $a = \pm\frac{1}{2}$, $\rho(a) = 0$. For $-\frac{1}{2} < a < \frac{1}{2}$, the defining infimum is obtained with $t = \tau(a)$, $\rho(0) = 1$ and for $a \neq 0$, $\rho(a) = e^{-a\tau}(2/\tau)\sinh(\tau/2)$.

PROOF. Routine calculus.

REMARK. Exponential centering of the probability density function $\chi_{[-1/2-a,1/2-a]}(x)$ yields the probability density function

$$f_a(x) = (1/\rho(a))e^{x\tau(a)}\chi_{[-1/2-a,1/2-a]}(x),$$

which is centered in the sense that $\int x f_a(x)\,dx = 0$.

(5.7) For $-\frac{1}{2} < a < \frac{1}{2}$, $d\rho/da = -\rho(a)\tau(a)$.

PROOF. Routine from (5.3) and (5.6) except at 0. Since ρ is even we just need $\lim_{a\to 0^+}\rho(a) = 1$ and $\lim_{a\to 0^+} d\rho/da = 0$. The first follows from (5.6) and from $(2/\tau)\sinh(\tau/2) \to 1$ as $\tau \to 0$. For the other, $\lim_{a\to 0^+}\rho(a) = 1$ and $\lim_{a\to 0^+}\tau(a) = 0$ so $\lim_{a\to 0^+} d\rho/da = \lim_{a\to 0^+} -\rho(a)\tau(a) = 0$. □

(5.8) $\lim_{a\to 1/2}\rho(a) = 0$ and $\lim_{a\to 1/2}\rho(a)\tau(a) = e$.

PROOF. For $0 < a < \frac{1}{2}$, $\rho(a) \leq \int_{-1/2-a}^{1/2-a} e^{(1/2-a)^{-1}x}\,dx$ by definition, so $\rho(a) \leq \int_{-\infty}^{(1/2-a)} e^{(1/2-a)^{-1}x}\,dx = (\frac{1}{2} - a)e$. For the second claim, $\rho\tau = 2e^{-a\tau}\sinh(\tau/2)$ from (5.6) and $a\tau = (\frac{1}{2} - 1/\tau + O(e^{-\tau}))\tau$ as $\tau \to \infty$ from (5.1). Since $\sinh(\tau/2) = \frac{1}{2}e^{\tau/2}(1 + O(e^{-\tau}))$,

$$\rho\tau = \exp(1 + O(\tau e^{-\tau}))(1 + O(e^{-\tau})) \to e \quad \text{as } \tau \to \infty.$$

Thus ρ is continuous and differentiable on $[-\frac{1}{2}, \frac{1}{2}]$, even, decreasing on $[0, \frac{1}{2}]$ and positive except that it is 0 at $\pm\frac{1}{2}$. From (5.7) for $-\frac{1}{2} < a < \frac{1}{2}$,

$$d^2\rho/da^2 = -(\tau^2 + d\tau/da)\rho < 0$$

since $d\tau/da > 0$. Since $d\rho/da$ is decreasing, $-e \leqslant d\rho/da \leqslant e$ for $-\frac{1}{2} \leqslant a \leqslant \frac{1}{2}$, so $(\frac{1}{2} - |a|)^{-1}\rho(a) < e$ on $(-\frac{1}{2}, \frac{1}{2})$. Since ρ is concave, $\rho(a) \geqslant 2(\frac{1}{2} - |a|)$ on $[-\frac{1}{2}, \frac{1}{2}]$, with equality at $\pm \frac{1}{2}$ and at 0.

Now let $\sigma^2 = \sigma^2(a) = \frac{1}{\rho}\int_{-1/2-a}^{1/2-a} x^2 e^{\tau x}\,dx = \int_{-\infty}^{\infty} f_a(x)\,dx$, for $-\frac{1}{2} < a < \frac{1}{2}$, and let $\sigma(\pm\frac{1}{2}) = 0$. Then σ is positive on $(-\frac{1}{2}, \frac{1}{2})$ and $\sigma(0) = 12^{-1/2}$.

(5.9) On $[0, \frac{1}{2}]$, σ is a decreasing function of a, $\sigma < 1/\tau$, $\lim_{a\to 1/2} \sigma\tau = 1$ and $\lim_{a\to 1/2} \sigma(a) = 0$.

PROOF. Evaluating the defining integral for σ and using (5.6),

$$\sigma^2 = \left(\frac{1}{4} + a^2 - \frac{a\cosh(\tau/2)}{\sinh(\tau/2)}\right) \quad \text{for } a \neq 0.$$

As $a \to 0$, $\tau \sim 12a$ so $\lim_{a\to 0} \sigma^2 = \frac{1}{12}$. Since

$$d(\sigma^2)/d\tau = -2\tau^{-3} + \tfrac{1}{4}\cosh(\tau/2)\sinh^{-3}(\tau/2) = d^2a/d\tau^2 < 0$$

for $0 < a < \frac{1}{2}$, σ is decreasing as a function of τ on $[0, \infty)$ and as a function of a on $[0, \frac{1}{2}]$. Integrating $d(\sigma^2)/d\tau = d^2a/d\tau^2$ gives $\sigma^2 = \tau^{-2} - (4\sinh^2(\tau/2))^{-1}$ so $\lim_{\tau\to\infty} \sigma^2 = \lim_{\tau\to\infty}(da/d\tau) = 0$, and since $\lim_{\tau\to\infty} \tau^2 da/d\tau = 1$ and $\tau^2 da/d\tau < 1$ on $[0, \infty)$, $\lim_{\tau\to\infty} \sigma\tau = 1$ and $\sigma < 1/\tau$ on $(0, \infty)$. Equivalently $\sigma(a) < 1/\tau(a)$ on $(0, \frac{1}{2})$ and $\lim_{a\to 1/2} \sigma(a)\tau(a) = 1$ so that $\lim_{a\to 1/2} \sigma(a) = 0$.

Now let $\beta = \beta(a) = \frac{1}{\rho}\int_{-1/2-a}^{1/2-a} |x|^3 e^{\tau x}\,dx$. Then β is even and $\beta(0) = \frac{1}{32}$.

(5.10) $\lim_{a\to 1/2} \tau^3\beta = (12e^{-1} - 2) > 0$.

PROOF. Equivalently $\lim \tau^3\beta = 12e^{-1} - 2$. Now

$$\tau^3\beta = \frac{1}{\rho\tau}\int_{(-1/2-a)}^{(1/2-a)} |y|^3 e^y\,dy.$$

Since $\lim_{\tau\to\infty} \rho\tau = e$, $\lim_{\tau\to\infty}(\frac{1}{2} - a)\tau = 1$ and $\lim_{\tau\to\infty}(\frac{1}{2} + a)\tau = \infty$, it follows that

$$\lim_{\tau\to\infty} \tau^3\beta = \frac{1}{e}\int_{-\infty}^{1} |y|^3 e^y\,dy = (12e^{-1} - 2) \cong 2.41 > 0.$$

Since β and $\tau^3\beta$ are bounded on any interval $[0, \frac{1}{2} - \varepsilon]$ as functions of a, there exists $C_7 > 0$ such that $\beta < C_7$ and $|\tau^3|\beta < C_7$ for $-\frac{1}{2} < a < \frac{1}{2}$.

REMARK. To use exponential centering we must estimate $\text{Prob}(\Sigma_1^N Z_i \geqslant 0)$, where the Z_i are N independent random variables each with density $f_a(x)$. The Berry-Esseen inequality, or more precisely some of Zolotarev's elaborations, permit this. The parameters β and σ are needed to find the Liapunov ratio (see §6). And ρ gives us the connection between probabilities for the Z_i and the original Y_i.

6. A uniform Chernoff's theorem. Recall that Y_n, $1 \leqslant n \leqslant N$, are independent random variables each with density $\chi_{[-1/2,1/2]}(x)$.

THEOREM 6.1. *There exists $C_8 > 0$ such that for $N \geqslant 1$ and $-\frac{1}{2} < a < \frac{1}{2}$,*

$$C_8 N^{-1/2}(\rho(a))^N \leqslant \text{Prob}\left(\left|\sum_1^N Y_i\right| \geqslant N|a|\right) \leqslant (\rho(a))^N.$$

PROOF. It is sufficient to show that there is $C_8 > 0$ such that if $N \geqslant 1$ and $0 < a < \frac{1}{2}$ then

$$C_8 N^{-1/2}(\rho(a))^N \leqslant \operatorname{Prob}\left(\sum_1^N Y_i \geqslant Na\right),$$

as the other inequality is an instance of Bernstein's result **[1]**, and as symmetry accounts for the other cases.

Let Z_n, $1 \leqslant n \leqslant N$, be independent, each with density

$$f_a(x) = \frac{1}{\rho(a)} e^{x\tau(a)} \chi_{[-1/2-a, 1/2-a]}(x).$$

Let $G_N(x)$ be the distribution function of $\Sigma_1^N Z_i$, and $H_N(x) = G_N(\sigma\sqrt{N}x)$. The Z_i have variance $\sigma^2 = \sigma^2(a)$. Let $\Phi(x) = \int_{-\infty}^x (2\pi)^{-1/2} e^{-u^2/2}\, du$ be the $\mathfrak{N}(0,1)$ distribution function.

LEMMA 6.2. *There exists $C_9 > 0$ such that if $0 \leqslant a < \frac{1}{2}$, $0 \leqslant x \leqslant 1$ and $N \geqslant 1$ then $H_N(x) - H_N(0) \geqslant C_9 x$.*

Before the proof we show how Theorem 6.1 follows from this lemma. Let $F_N(x)$ be the distribution function of $\Sigma_1^N(Y_i - a)$, and let $P_1(a, N) = \operatorname{Prob}(\Sigma_1^N Y_i \geqslant Na)$. ($P(a, N)$ of §5 and $P_1(a, N)$ are equal to within a factor of 2). Following Bahadur **[1]**,

$$\begin{aligned} P_1(a, N) &= \int_0^\infty dF_N(y) = \rho^N \int_0^\infty e^{-\tau y}\, dG_N(y) \\ &\geqslant \rho^N \int_0^{\sigma\sqrt{N}x} e^{-\tau y}\, dG_N(y) \geqslant \rho^N e^{-\tau\sigma\sqrt{N}x} \int_0^{\sigma\sqrt{N}x} dG_N(y) \\ &= \rho^N e^{-\tau\sigma\sqrt{N}x}(H_N(x) - H_N(0)) \geqslant C_9 x \rho^N e^{-\tau\sigma\sqrt{N}x} \end{aligned}$$

for $0 \leqslant x \leqslant 1$.

Now if $\tau\sigma\sqrt{N} < 1$ we take $x = 1$, and then $C_9 x\rho^N e^{-\tau\sigma\sqrt{N}} \leqslant C_9 e^{-1}\rho^N$. Otherwise we take $x = (\tau\sigma\sqrt{N})^{-1}$. Since $\tau\sigma$ is bounded above by some $C_{10} > 0$ for $0 < a < \frac{1}{2}$, we have an absolute constant $C_{11} > 0$ such that $P(a, N) \geqslant C_{11} N^{-1/2}\rho^N$ for $0 \leqslant a < \frac{1}{2}$ and $N \geqslant 1$. □

We now prove Lemma 6.2. By Theorem 4 of [7], for $N \geqslant 1$, x any real number and $0 \leqslant a < \frac{1}{2}$,

$$|H_N(x) - \Phi(x)| \leqslant (.68705) N^{-1/2}\beta\sigma^{-3}.$$

From (5.9) and the points made after (5.10) about $\tau^3\beta$, there is $C_{12} > 0$ such that if $0 \leqslant a < \frac{1}{2}$ then $\beta\sigma^{-3} < C_{12}$. Thus there is $C_{13} > 0$ such that if $0 \leqslant a < \frac{1}{2}$ and $N \geqslant 1$ then $|H_N(x) - \Phi(x)| \leqslant C_{13} N^{-1/2}$. In particular there exists $M > 0$ such that if $0 \leqslant a < \frac{1}{2}$ and $N \geqslant M$ then $|H_N(x) - \Phi(x)| \leqslant \frac{1}{100}$ for each of $x = -\frac{1}{2}, 0, 1$ and $\frac{3}{2}$.

Let $h_N(x) = dH_N/dx$. Then for each $N \geqslant M$ and each a, $0 \leqslant a < \frac{1}{2}$, there exists $x_1 = x_1(N, a) \in [-\frac{1}{2}, 0]$ and $x_2 \in [1, \frac{3}{2}]$ such that $h_N(x_1) \geqslant \frac{1}{10}$ and $h_N(x_2) \geqslant \frac{1}{10}$. Since $h_N(x)$ is the N-fold convolution of $f_a(x)$ which is log concave, $h_N(x)$ is log concave and thus unimodal. Consequently if $N \geqslant M$, $0 \leqslant a < \frac{1}{2}$ and $0 \leqslant x \leqslant 1$ then $h_N(x) \geqslant \frac{1}{10}$.

If $N \leqslant M$ we observe that $f_a(x) \geqslant e^{(a-1/2)\tau}\chi_{[a-1/2,1/2-a]}(x)$. Let $\lambda_C(x) = (2C)^{-1}\chi_{[-C,C]}(x)$. Since $(\frac{1}{2}-a)\tau$ is bounded above on $0 \leqslant a < \frac{1}{2}$, there exists $C_{14} > 0$ such that $f_a(x) \geqslant C_{14}\lambda_{(1/2-a)}(x)$. Let Λ_i, $1 \leqslant i \leqslant M$, be independent random variables each with density $\lambda_{(1/2-a)}(x)$. Then

$$H_N(x) - H_N(0) \geqslant C_{14}^M \min_{N \leqslant M} \operatorname{Prob}\left(\sum_1^N \Lambda_i \in \left[0, \sigma\sqrt{N}\,x\right]\right)$$

$$\geqslant C_{14}^M x \operatorname{Prob}\left(\sum_1^M \Lambda_i \in [0, \sigma]\right) \geqslant C_{15}x,$$

since $\sigma \sim (\frac{1}{2}-a)$ as $a \to \frac{1}{2}$ and $(\frac{1}{2}-a)$ is the radius of the interval on which Λ_i is supported. We had already shown $h_N(x) \geqslant \frac{1}{10}$ for $N \geqslant M$ so that in this case also $H_N(x) - H_N(0) \geqslant \frac{1}{10}x$. We take $C_9 = \min(\frac{1}{10}, C_{15})$ which proves Lemma 6.2. □

7. The factorization decline function $f(r)$. As we shall see in §8, the probability in Corollary 4.2, if $E = [h\log^{1/2}x]$, is $\operatorname{Prob}(t_1 \leqslant \Sigma_1^E X_i \leqslant t_2)$ and is estimated[1] to within powers of $\log x$ by $\operatorname{Prob}(|\Sigma_1^E Y_i| \geqslant E\,|\,a\,|)$ where $a = (\frac{1}{2} - \frac{r}{h})$ and $r = \log^{1/2}x\log^{-1}y$. Also, the distinction between $(\log y - \log z)^E$ and $(\log y)^E$ is negligible, that is, $|E\log(1-\log z/\log y)| = O(rh\log_2 x)$. Thus from (4.4) $\log K(x,y) \geqslant \log x + E\log_2 y - E\log E + E + E\log\rho(\frac{1}{2}-\frac{r}{h}) + O((1+rh)\log_2 x)$, or equivalently

$$\text{(7.0)}\quad \log K(x,y) \geqslant \log x + \log^{1/2}x\left(-h\log r - h\log h + h + h\log\rho\left(\tfrac{1}{2}-\tfrac{r}{h}\right)\right) + O\left((1+rh)\log_2 x\right).$$

To make this plan go we need two things. We need to supply details for the various estimates connecting (4.4) to (7.0), and we need to know more about $g(h,r) = (-h\log r - h\log h + h + h\log\rho)$, in particular how to choose $h = h(r)$ to maximize $g(h,r)$. This second requirement motivates the definitions and calculations that follow.

Let $g(h,r) = (-h\log r - h\log h + h + h\log\rho(\frac{1}{2}-\frac{r}{h}))$ for $h > r$, and $-\infty$ otherwise. Let $f(r) = \sup_{h>r} g(h,r)$. (This is the $f(r)$ of §4.) We prove

(7.1) $f(r)$ is a decreasing function of r on $(0,\infty)$.

(7.2) $\lim_{r\to 0} f(r) = 2$.

(7.3) $f(r)$ is differentiable and concave down on $(0,\infty)$.

(7.4) There is a unique $h = h(r) > r$ such that $f(r) = g(h(r), r)$.

(7.5) $\lim_{r\to 0} h(r) = 1$, $h(r)$ is differentiable and increasing on $(0,\infty)$ and $h(r)/r$ is decreasing on $(0,\infty)$.

(7.6) $\lim_{r\to\infty} h(r)/r = 1$, and $\inf_{r>0}(h(r) - r) > 0$.

We conclude the section with a short table of $h(r)$ and $f(r)$ generated on a pocket computer.[2]

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[1] There are conditions limiting r and h in terms of x on the estimation of $P_E(t_1, t_2)$ by ρ^E but we defer discussion of this.

[2] The program is in BASIC and is available on request. No attempt has been made at formal program verification. The rest of the paper is independent of this table.

First we prove (7.4). From $g(h, r) = -h\log r - h\log h + h + h\log\rho$ with $\rho = \rho(\frac{1}{2} - \frac{r}{h})$ we have

(7.7) $\partial g/\partial h = -\log r - \log h + \log\rho - r\tau/h$, and

(7.8) $\partial^2 g/\partial h^2 = -1/h - (r^2/h^3)\, d\tau/da < 0$.

Since $\lim_{h\to r^+} g(h, r) = \lim_{h\to\infty} g(h, r) = -\infty$, $g(h, r)$ is concave as a function of h on (r, ∞) and drops at both ends. Thus there is a unique $h = h(r)$ such that $\partial g/\partial h = 0$, and for that h, $g(h, r) = f(r)$.

To prove (7.5) we show $dh/dr > 0$. Since $\partial g/\partial h = 0$ when $h = h(r)$,

(7.9) $-\log r - \log h + \log\rho - r\tau/h = 0$ when $h = h(r)$.

For $r > 0$, (7.9) defines $h(r)$ as an implicit function of r. From (7.9) we calculate (writing h for $h(r)$) that

$$dh/dr = \frac{h}{r}\left(\frac{r^2}{h^2}\frac{d\tau}{da} - 1\right)\left(\frac{r^2}{h^2}\frac{d\tau}{da} + 1\right)^{-1} \tag{7.10}$$

where $\tau = \tau(a)$ and $a = (\frac{1}{2} - \frac{r}{h})$. We claim $dh/dr > 0$.

Since $d\tau/da > 0$ for all a in $(-\frac{1}{2}, \frac{1}{2})$, this will follow from $(r^2/h^2)d\tau/da > 1$, or equivalently, from $d\tau/da > (\frac{1}{2} - a)^{-2}$, i.e. $da/d\tau < (\frac{1}{2} - a)^2$.

In §5 we found

$$\frac{da}{d\tau} = \frac{1}{\tau^2} - \frac{1}{4\sinh^2(\tau/2)} \quad \text{and} \quad \left(\frac{1}{2} - a\right) = \left(\frac{1}{2} + \frac{1}{\tau} - \frac{\cosh(\tau/2)}{2\sinh(\tau/2)}\right).$$

Thus we need

$$\frac{1}{\tau^2} - \frac{1}{4\sinh^2(\tau/2)} < \left(\frac{1}{2} + \frac{1}{\tau} - \frac{\cosh(\tau/2)}{2\sinh(\tau/2)}\right)^2 \tag{7.11}$$

for all $\tau \neq 0$. (We know $da(0)/d\tau = \frac{1}{12} < (\frac{1}{2} - 0)^2$.) Let $s = \tau/2$, square out the right side and simplify (7.11) to get

$$\frac{\cosh(s)}{\sinh(s)}\left(1 + \frac{1}{s}\right) < 1 + \frac{1}{s} + \frac{1}{\sinh^2(s)}. \tag{7.12}$$

If $s > 0$ we multiply both sides of (7.12) by s. Then squaring both sides, using $\cosh^2 = 1 + \sinh^2$ and simplifying, one gets $1 < s^2 + s^2/\sinh^2(s)$, which follows from $\sinh(s) < s\cosh(s)$ for $s > 0$.

If $s < 0$, we multiply (7.12) by $s\sinh(s) > 0$ and let $v = -s$. Then (7.12) becomes

$$(1 - v)\cosh v < (v - 1)\sinh v + v/\sinh(v) \quad \text{for } v > 0. \tag{7.13}$$

If $v \geqslant 1$ the left side of (7.13) is $\leqslant 0$ while the right side is positive. If $0 < v < 1$ then (7.12) is equivalent to

$$\frac{\cosh v}{\sinh v}\left(\frac{1}{v} - 1\right) < \left(1 - \frac{1}{v} + \frac{1}{\sinh^2 v}\right). \tag{7.14}$$

Both sides of (7.14) are continuous on $(0, 1]$, and the right side is positive at 1 while the left side is positive on $(0, 1)$. If the right side were negative anywhere on $(0, 1)$ (in fact it is not), it would be zero somewhere on $(0, 1)$. Thus if $(\text{right})^2 > (\text{left})^2 > 0$ on $(0, 1)$ then $(\text{right}) > 0$ on $(0, 1)$ so $(\text{right}) > (\text{left})$ on $(0, 1)$. Thus squaring (7.14) and simplifying we reduce $(7.14)^2$ successively to $(\frac{1}{v} - 1)^2 < -2(\frac{1}{v} - 1) + 1/\sinh^2 v$, to $\frac{1}{v} < \cosh v/\sinh v$, and finally to $\sinh v < v\cosh v$ for $v > 0$. This proves (7.14), (7.12), (7.11) and the claim that $dh/dr > 0$.

We now prove $\lim_{r\to 0^+} h(r) = 1$. For fixed $h > 0$,

$$\begin{aligned}\lim_{r\to 0^+} \partial g/\partial h &= \lim_{r\to 0^+} (-\log r - \log h + \log \rho - r\tau/h) \\ &= \lim_{r\to 0^+} (-\log r - \log h + \log(er/h) - 1) \quad \text{from (5.5) and (5.8)} \\ &= -2\log h.\end{aligned}$$

Let $1 > \varepsilon > 0$. For r sufficiently small, if $h \leqslant 1 - \varepsilon$ then $\partial g/\partial h > 0$, and if $h \geqslant 1 + \varepsilon$ then $\partial g/\partial h < 0$. Since $\partial g/\partial h$ is decreasing on (r, ∞), the value $h = h(r)$ for which $\partial g/\partial h = 0$ lies in $(1 - \varepsilon, 1 + \varepsilon)$ for r sufficiently small. That is, $\lim_{r\to 0^+} h(r) = 1$.

Since $dh/dr > 0$, $h(r)$ is increasing and greater than 1 on $(0, \infty)$. To show $h(r)/r$ decreasing we note $dh/dr < h/r$, which follows from (7.10). This completes the proof of (7.5)

Also from (7.10) if $\lim_{r\to\infty}(h(r)/r) = C > 1$ (the limit exists since $h(r) > r$ and $h(r)/r$ is decreasing) then

$$\lim_{r\to\infty} (dh/dr) = C\left(\frac{L}{C^2} - 1\right)\left(\frac{L}{C^2} + 1\right)^{-1} < C,$$

where $L = d\tau/da$ at $(\frac{1}{2} - \frac{1}{C})$. Then by L'Hôpital's rule $\lim_{r\to\infty}(h/r) < C$, a contradiction. Thus $\lim_{r\to\infty}(h(r)/r) = 1$. The other claim in (7.6) follows from $dh/dr > 1$ for large r. As $r \to \infty$, $h/r \to 1$ and $a \to -\frac{1}{2}$. Now from (7.10), $dh/dr > 1$ is equivalent to

$$\left(\frac{1}{2} - a\right)^2 \frac{d\tau}{da} - 1 > \left(\frac{1}{2} - a\right)^3 \frac{d\tau}{da} + \left(\frac{1}{2} - a\right),$$

or

$$\left(\frac{1}{2} - a\right)^2\left(\frac{1}{2} + a\right) > \left(\frac{3}{2} - a\right)\frac{da}{d\tau}.$$

Now $(\frac{1}{2} + a) = -\frac{1}{\tau} + O(e^{\tau/2})$, $(\frac{1}{2} - a) \to 1$, $(\frac{3}{2} - a) \to 2$ and $da/d\tau \sim 1/\tau^2$ so as $a \to -\frac{1}{2}$, $\tau \to -\infty$ and $-\frac{1}{\tau} > 1/\tau^2$ with a wide margin for the error terms.

Now $f(r) = g(h(r), r)$ so $df/dr = \partial g/\partial r + (\partial g/\partial h)dh/dr$. But at $h = h(r)$, $\partial g/\partial h = 0$ so $df/dr = \partial g/\partial r = -h/r + \tau$. Since $a = \frac{1}{2} - \frac{r}{h}$ and $(\frac{1}{2} - a)\tau < 1$ from §5, $df/dr < 0$ for $0 < a < \frac{1}{2}$. At $a = 0$ ($h(r) = 2r$), $df/dr = -2$. For $a < 0$, $-h/r$ and τ are both negative so again $df/dr < 0$. This proves (7.1).

As $r \to 0$, $h(r) \to 1$, $r/h \to 0$ and $a \to \frac{1}{2}$. From $a = \cosh(\tau/2)/2\sinh(\tau/2) - \frac{1}{\tau}$ we have $(\frac{1}{2} - a) = \frac{1}{\tau} - 1/(e^\tau - 1)$ so $df/dr = -\tau^2/(e^\tau - \tau - 1)$ (which gives another proof that $df/dr < 0$), and so $\lim_{r\to 0^+} df/dr = 0$. In fact $df/dr = -r^2e^{-1/r}e^{O(1)}$ as $r \to 0$. Pursuing this would yield an asymptotic expansion for $f(r)$ at 0 but we have no need for one.

We now prove $\lim_{r\to 0^+} f(r) = 2$. For $\varepsilon > 0$ there exists δ, $0 < \delta < 1$, such that if $0 < r < \delta$ and $1 < h < 1 + \delta$ then $|\partial g/\partial h| < \varepsilon$ and $h(r) < 1 + \delta$. Thus for $0 < r < \delta$, $|f(r) - g(1, r)| < \varepsilon\delta < \varepsilon$. Now $\lim_{r\to 0^+} g(1, r) = \lim_{r\to 0^+}(-\log r + 1 + \log(er))$ $= 2$. Thus $\lim_{r\to 0^+} g(h(r), r) = 2$.

To prove f concave down we show that df/dr is decreasing. Since $h(r)/r$ is decreasing as a function of r, so is $a = (\frac{1}{2} - r/h(r))$. Thus if $df/dr = (\tau - (\frac{1}{2} - a)^{-1})$ is an increasing function of a or of τ, it is decreasing as a function of r. Differentiating $(\tau - (\frac{1}{2} - a)^{-1})$ with respect to τ gives $(\frac{1}{2} - a)^{-2}(-da/d\tau) + 1$ and this is positive since from (7.10), $(\frac{1}{2} - a)^2 > da/d\tau$.

	r	$h(r)$	$f(r)$	r	$h(r)$	$f(r)$
	0	1	2			
	0.1	1.000205	1.999955	1.2	1.93385	0.134272
	0.2	1.015247	1.992804	1.4	2.15277	−0.522509
	0.3	1.062838	1.955752	1.6	2.37376	−1.24236
	0.4	1.133229	1.878847	1.8	2.59607	−2.01777
	0.5	1.217162	1.762951	2.0	2.81922	−2.84278
	0.6	1.309432	1.611562	4.0	5.06244	−13.05606
	0.7	1.407116	1.428356	6.0	7.30033	−25.6159
	0.8	1.508478	1.216610	8.0	9.52831	−39.6936
	0.9	1.612437	0.979127	10.0	11.7478	−54.8951
	1	1.718282	0.718282	100	109.289	−1067.1
				1000	1063.73	−15628.5
(In fact	1	$e - 1$	$e - 2$)	10000	10486.9	−204893
				100000	103947	−2529600

8. The lower bound for $K(x, y)$.

THEOREM 8.1. *There exists a real constant C_{16} such that if $r = \sqrt{\log x}/\log y$ and if $(4 \log\log x)^{-1} \leqslant r \leqslant \frac{1}{12}\sqrt{\log x}/\log\log x$, then $\log K(x, y) \geqslant \log x + f(r)\sqrt{\log x} - C_{16}((1 + r)^2 \log\log x)$.*

PROOF. The idea is to use (4.4) with $h = h(r)$ and $E = [h \log^{1/2} x]$, replace the random variables X_i on $\{A + 1, \ldots, B\}$ with Y_i on $[-\frac{1}{2}, \frac{1}{2}]$ and use Theorem 6.1. When $\Sigma_1^E X_i$ is near $\log x/\log\alpha$, $\Sigma_1^E Y_i$ is close to $E(\frac{r}{h} - \frac{1}{2})$. So if we take logs in (4.4), $E \log(\log y - \log z)$ is roughly $h \log^{1/2} x(\frac{1}{2}\log_2 x - \log r)$, $-E \log E + E$ is roughly $h \log^{1/2} x(-\frac{1}{2}\log_2 x - \log h + 1)$, and

$$\log \operatorname{Prob}\left(\sum_1^E X_i \in ([\log x/\log\alpha] - [\log_2 x/\log\alpha] + 1, [\log x/\log\alpha])\right)$$

is roughly $h \log^{1/2} x \log \rho(\frac{1}{2} - \frac{r}{h})$. These add to $(-h \log r - h \log h + h + h \log \rho) \log^{1/2} x = f(r)\sqrt{\log x}$. To make this rigorous we need some estimates. Before getting into the details of the proof we note that if $r < (4 \log_2 x)^{-1}$ then Theorem 3.1 applies and $K(x, y) \sim K(x)$, while if $r > \frac{1}{12}\log^{1/2} x \log_2^{-1} x$, Theorem 2.1 gives a fairly good estimate of $K(x, y)$.

The variables X_n, $1 \leqslant n \leqslant E$, are integer valued and equally distributed on $\{A + 1, \ldots, B\}$. Let W_n, $1 \leqslant n \leqslant E$, be a further E random variables each with density $\chi_{[0,1]}(s)$, such that $\{X_1, \ldots, X_E, W_1, \ldots, W_E\}$ are independent. Let

$$Y_i = (B - A)^{-1}\left(X_i - W_i - \tfrac{1}{2}(B + A)\right).$$

Then $\{Y_1,\dots,Y_E\}$ are independent and identically distributed with density $\chi_{[-1/2,1/2]}(s)$. Thus if $\Sigma_1^E X_i = S$,

$$(S-EA-E)(B-A)^{-1}-\frac{1}{2}E\leqslant\sum_1^E Y_i\leqslant(S-EA)(B-A)^{-1}-\frac{1}{2}E.$$

So

$$\operatorname{Prob}\left(T-V\leqslant\sum_1^E X_i\leqslant T\right)\geqslant\operatorname{Prob}\left((T-V-EA)(B-A)^{-1}-\frac{1}{2}E\leqslant\sum_1^E Y_i\right.$$
$$\left.\leqslant(T-EA-E)(B-A)^{-1}-\frac{1}{2}E\right).$$

For any real q_1, q_2 with $-\frac{1}{2}E\leqslant q_1<q_2\leqslant\frac{1}{2}E$ let $b=b(q_1,q_2)=q_1$ if $|q_1|<|q_2|$, q_2 otherwise. The probability density function of $\Sigma_1^E Y_i$ is the convolution of E copies of $\chi_{[-1/2,1/2]}(s)$ so it is symmetric and unimodal. Thus for any q_1 and q_2 as above,

$$\operatorname{Prob}\left(\sum_1^E Y_i\in[q_1,q_2]\right)\geqslant\frac{1}{2}(q_2-q_1)E^{-1}\operatorname{Prob}\left(\left|\sum_1^E Y_i\right|\geqslant|b|\right).$$

Now in the application x and $r=\log^{1/2}x\log^{-1}y$ are given, $h=h(r)$, $E=[h\log^{1/2}x]$, $T=[\log x/\log\alpha]$ and $V=[\log_2 x/\log\alpha]-1$. (If $\Sigma_1^E X_i=S\in[T-V,T]$ then $x/\log x\leqslant\alpha^S\leqslant x$.)

Let $q_1=(T-V-EA)(B-A)^{-1}-\frac{1}{2}E$ and $q_2=(T-EA-E)(B-A)^{-1}-\frac{1}{2}E$. If $\Sigma_1^E Y_i\in[q_1,q_2]$ then $\Sigma_1^E X_i\in[T-V,T]$ so

$$\operatorname{Prob}\left(\sum_1^E X_i\in[T-V,T]\right)\geqslant\operatorname{Prob}\left(\sum_1^E Y_i\in[q_1,q_2]\right)$$
$$\geqslant\frac{1}{2}(q_2-q_1)E^{-1}\operatorname{Prob}\left(\left|\sum_1^E Y_i\right|\geqslant|b|\right).$$

Since $q_2-q_1\sim\log_2 x/\log y>1/\log x$ and since $E\leqslant\log x$, this last is $\geqslant\log^{-2}x\operatorname{Prob}(|\Sigma_1^E Y_i|\geqslant|b|)$. Now

$$\begin{aligned}q_2&=-\tfrac{1}{2}E+(\log x/\log y)\big(1-h(1+\log z)\log^{-1/2}x+O(\log z/\log x)\big)\\&\quad\cdot\big(1-r\log z\log^{-1/2}x\big)^{-1}\\&=-\tfrac{1}{2}E+r\log^{1/2}x\big(1-h\log^{-1/2}x(1+\log z)+O\big(\log^{-1/2}x\big)\big)\\&\quad\cdot\big(1-r\log z\log^{-1/2}x\big)^{-1},\end{aligned}$$

so for $r<\frac{1}{12}\log^{1/2}x\log_2^{-1}x$, $q_2/E=-\frac{1}{2}+\frac{r}{h}(1-O((h-r)\log z+h)\log^{-1/2}x)=-\frac{1}{2}+\frac{r}{h}+o(r\log_2 x\log^{-1/2}x)$.

Now $q_2-q_1\sim r\log_2 x\log^{-1/2}x$ so q_1 and b are also $-\frac{1}{2}+\frac{r}{h}+o(r\log_2 x\log^{-1/2}x)$. We now estimate the effect of replacing $a=(-\frac{1}{2}+\frac{r}{h})$ with $a'=b/E=(-\frac{1}{2}+\frac{r}{h}+o(r\log_2 x\log^{-1/2}x))$. For small r, say $r<\frac{1}{3}$, $0>a>a'$ so $\rho(a)>\rho(a')$. But $d\log\rho/da=-\tau(a)$ so $\log\rho(a)-\log\rho(a')<(a-a')|\tau(a')|<4(a-a')|\tau(a)|$ because $2(a'+\frac{1}{2})>(a+\frac{1}{2})$ so that τ cannot increase by any large factor between a and a'. The claim $2(a' + \frac{1}{2}) > (a + \frac{1}{2})$ amounts to $2(r/h) + o(r \log_2 x \log^{-1/2} x) > r/h$, and since h is bounded for $0 < r \leqslant \frac{1}{3}$ this holds for x sufficiently large. Now $|\tau(a)| \leqslant h/r$ and $(a - a') = o(r \log_2 x \log^{-1/2} x)$ so $\log \rho(a) - \log \rho(a') = o(h \log_2 x \log^{-1/2} x)$. For r sufficiently large ($r > C_{17}$), $0 < a' < a$ so $\log \rho(a) < \log \rho(a')$. In the intermediate interval $(\frac{1}{3}, C_{17})$, $|\tau|$ is bounded so again $\log \rho(a) - \log \rho(a') = o(h \log_2 x \log^{-1/2} x)$. Thus in any case $\log \rho(a) - \log \rho(a') \leqslant o(h \log_2 x \log^{-1/2} x)$, so

$$h \log^{1/2} x \log \rho(a) - E \log \rho(a') \leqslant o\big(h^2 \log_2 x\big).$$

Since $h = O(1 + r)$, this proves that

$$\log \operatorname{Prob}\left(\sum_1^E X_i \in (T - V, T) \right) \geqslant \log C_8 - 3 \log_2 x - o\big((1 + r)^2 \log_2 x\big) + h \log^{1/2} x \log \rho(a).$$

Now from (4.4) and since $E \leqslant \log x$,

$$\begin{aligned} \log K(x, y) &\geqslant \log x - O(\log_2 x) - o\big((1 + r)^2 \log_2 x\big) + E \log(\log y - \log z) \\ &\quad - E \log E + E + h \log^{1/2} x \log \rho \\ &= \log x - o\big((1 + r)^2 \log_2 x\big) - O(\log_2 x) \\ &\quad + \log^{1/2} x(-h \log r - h \log h + h + h \log \rho) \quad (\text{here is } f(r)) \\ &\quad + O(\log_2 x) + O(\log r) + O(hr \log_2 x) + O(\log h) \\ &\quad + O(\log_2 x) + O(1). \end{aligned}$$

We have $r > (4 \log_2 x)^{-1}$ and $r < \frac{1}{12} \log^{1/2} x \log_2^{-1} x$, and $\rho > r/h > \frac{1}{2} r$ as $r \to 0$, while $\rho > (1 - r/h) > C_{17}/r$ as $r \to \infty$ from (7.6). Thus the error terms above reduce to $O((1 + r)^2 \log_2 x)$ and we have

$$\log K(x, y) \geqslant \log x + f(r) \sqrt{\log x} - C_{16}(1 + r)^2 \log_2 x. \quad \square$$

9. Upper bounds. We keep the same notation with the minor difference that the random variables X_i are now on $\{A, A + 1, \ldots, B - 1\}$.

THEOREM 9.1. *For* $r = \sqrt{\log x} / \log y$, $r \leqslant C_{19} \log^{1/4} x$, *we have*

$$K(x, y) \leqslant x \exp\big(f(r) \sqrt{\log x} + O\big((\log \log x)^2 + r^2 \log \log x\big)\big).$$

PROOF. There are two new issues here. First, we must consider all possible values of E and not just pick one out which seems to give a large contribution, and we must also consider the possibility that $\sum_1^E X_i$ is considerably less than $\log x / \log \alpha$. Second, the small divisors ($d \leqslant z$) cannot be dismissed out of hand. Their effect, nonetheless, remains insignificant. We begin with some estimates. Let $C = 1 + [\log z / \log 2]$. For $t \geqslant 1$ let $U(t) = \{(u_1, u_2, \ldots, u_C)$: all u_j are nonnegative integers, and $\sum_1^C j u_j \leqslant \log t / \log 2\}$. For real $s \geqslant 1$ and integer $E \geqslant 1$ let $V(s, E) = \{(v_A, \ldots, v_{B-1})$: all v_i are nonnegative integers, $\sum_A^{B-1} v_i = E$ and $\sum_A^{B-1} i v_i \log \alpha \leqslant \log s\}$. Let R_j denote the real interval $[2^j, 2^{j+1})$ for $1 \leqslant j \leqslant C$, and let W_j be the real interval $[\alpha^j, \alpha^{j+1})$ for $A \leqslant j \leqslant B - 1$. Let w_j be the number of integers in W_j. Then $w_j = \alpha^j(\alpha - 1) + O(1)$.

Let $S(x, y)$ be the set of all factorizations ξ, ξ: $\{2, 3, \ldots, y\} \to \{0, 1, 2, \ldots\}$ such that $\prod_2^y d^{\xi_d} \leqslant x$, so that $K(x, y) = \#S(x, y)$. For $\xi \in S(x, y)$ and $1 \leqslant j \leqslant C$ let u_j $(= u_j(\xi)) = \sum_{d \in R_j} \xi_d$, and for $A \leqslant j \leqslant B - 1$ let $v_j = \sum_{d \in W_j} \xi_d$. Let $\bar{u}(\xi) = (u_1, u_2, \ldots, u_C)$ and let $\bar{v}(\xi) = (v_A, \ldots, v_{B-1})$. Then for $\xi \in S(x, y)$.

$$\sum_1^C j u_j \log 2 + \sum_A^{B-1} j v_j \log \alpha \leqslant \log x. \tag{9.1}$$

The number of ξ, in $S(x, y)$ or not, having a given $\bar{u} = (u_1, \ldots, u_C)$ and $\bar{v} = (v_A, \ldots, v_{B-1})$ is

$$\prod_1^C \binom{2^{j-1} + u_j - 1}{u_j} \prod_A^{B-1} \binom{v_j + w_j - 1}{v_j}.$$

Now

$$\prod_A^{B-1} \binom{v_j + w_j - 1}{v_j} \leqslant \prod_A^{B-1} \left(w_j^{v_j}/v_j!\right) \exp\left(v_j^2/w_j\right), \tag{9.2}$$

and $w_j = (\alpha - 1)\alpha^j + O(1)$. Let $F(\bar{v}) = \sum_A^{B-1} v_j$, and $G(\bar{v}) = \sum_A^{B-1} j v_j$. Then

$$\prod_A^{B-1} \binom{v_j + w_j - 1}{v_j} \leqslant (\alpha - 1)^{F(\bar{v})} \alpha^{G(\bar{v})} \exp\left(\sum_A^{B-1} v_j^2/w_j\right) \cdot \left(1 + O\left(\log^{-2} x\right)\right)^{F(\bar{v})} \sum_A^{B-1} \left(1/v_j!\right). \tag{9.3}$$

Now for $\xi \in S(x, y)$, $F(\bar{v}(\xi)) \leqslant \log x/\log z$, so $(1 + O(\log^{-2} x))^{F(\bar{v})} = O(1)$. Also $\sum_A^{B-1} (v_j^2/w_j) \leqslant (F(\bar{v}))^2/w_1 = O(1)$ for $\xi \in S(x, y)$. Thus for $\xi \in S(x, y)$, there is $C_{18} > 0$ so that

$$\prod_A^{B-1} \binom{v_j + w_j - 1}{v_j} \leqslant C_{18}(\alpha - 1)^{F(\bar{v})} \alpha^{G(\bar{v})} \prod_A^{B-1} \left(1/v_j!\right). \tag{9.4}$$

For the "small divisors" we need not be so precise. For $\xi \in S(x, y)$, let $F_1(\bar{u}) = \sum_1^C u_j$ and $G_1(\bar{u}) = \sum_1^C j u_j$. Then

$$\prod_1^C \binom{u_j + 2^j - 1}{u_j} \leqslant 2^{G_1(\bar{u})} \prod_1^C \binom{u_j + 2^j - 1}{u_j} 2^{-j u_j} \leqslant 2^{G_1(\bar{u})}.$$

Thus if $\xi \in S(x, y)$,

$$\prod_1^C \binom{u_j + 2^j - 1}{u_j} \prod_A^{B-1} \binom{w_j + v_j - 1}{v_j} \leqslant C_{18}(\alpha - 1)^{F(\bar{v})} x \prod_A^{B-1} \left(1/v_j!\right). \tag{9.5}$$

For fixed $\bar{u}$ and $\bar{v}$ the number of $\xi \in S(x, y)$ such that $\bar{u}(\xi) = \bar{u}$ and $\bar{v}(\xi) = \bar{v}$ is thus $\leqslant C_{18}(\alpha - 1)^{F(\bar{v})} x \prod_A^{B-1} (1/v_j!)$. To count the number of $\bar{u}$ with $G_1(\bar{u}) \leqslant \log x/\log 2$ (so that if for some $\xi \in S(x, y)$, $\bar{u}(\xi) = \bar{u}$ then $\prod_1^C d^{\xi_d} \geqslant 2^{G_1(\bar{u})}$ need not exceed x), we introduce $N(R_1, R_2) = \#\{\bar{u}\colon \sum_1^{R_1} j u_j \leqslant R_2$ and the u_j's are nonnegative integers for $1 \leqslant j \leqslant R_1\}$.

LEMMA 9.2. *$N(R_1, R_2) \leqslant (R_1!)^{-2}(R_2 + R_1^2)^{R_1}$ for all integers R_1 and $R_2 \geqslant 1$.*

PROOF. $N(1, R_2) = R_2 + 1 = (1!)^{-2}(R_2 + 1^2)^1$. This starts an induction. Now

$$N(R_1, R_2) = \sum_{k=0}^{[R_2/R_1]} N(R_1 - 1, R_2 - kR_1)$$

$$\leqslant \sum_{0}^{[R_2/R_1]} ((R_1 - 1)!)^2 \left(R_2 - kR_1 + (R_1 - 1)^2\right)^{R_1 - 1}.$$

For any integers $m \geqslant 0$ and $R_1 \geqslant 2$,

$$m^{R_1 - 1} \leqslant \frac{1}{R_1} \int_{m - R_1/2}^{m + R_1/2} t^{R_1 - 1}\, dt.$$

Thus

$$N(R_1, R_2) \leqslant \sum_{k=0}^{[R_2/R_1]} ((R_1 - 1)!)^{-2} R_1^{-1} \int_{R_2 - kR_1 + (R_1 - 1)^2 - R_1/2}^{R_2 - kR_1 + (R_1 - 1)^2 + R_1/2} t^{R_1 - 1}\, dt$$

$$= (R_1!)^{-1}((R_1 - 1)!)^{-1} \int_{R_2 - R_1[R_2/R_1] + (R_1 - 1)^2 - R_1/2}^{R_2 + (R_1 - 1)^2 + R_1/2} t^{R_1 - 1}\, dt$$

$$\leqslant (R_1!)^{-1}((R_1 - 1)!)^{-1} \int_{(R_1 - 1)^2 - R_1/2}^{R_2 + (R_1 - 1)^2 + R_1/2} t^{R_1 - 1}\, dt$$

$$\leqslant (R_1!)^{-1}((R_1 - 1)!)^{-1} \int_{0}^{R_2 + R_1^2} t^{R_1 - 1}\, dt$$

$$= (R_1!)^{-2}\left(R_2 + R_1^2\right)^{R_1}. \quad \square$$

In particular $N(C, 1 + [\log x/\log 2]) \leqslant (C!)^{-2}(1 + C^2 + \log x/\log 2)^C$. Since $C = O(\log\log x)$, $N(C, 1 + [\log x/\log 2]) = \exp(O(\log\log x)^2)$ is an upper bound for the number of different $\bar{u}$ such that there is $\xi \in S(x, y)$ with $\bar{u}(\xi) = \bar{u}$.

We now estimate, for arbitrary fixed integer E, the quantity

$$Q(E) = (\alpha - 1)^E \sum_{\bar{v} \in V(x, E)} \prod_{A}^{B-1} (1/v_j!).$$

As in §4, $Q(E) = (\alpha - 1)^E (B - A)^E (E!)^{-1} \mathrm{Prob}(\sum_1^E X_i \leqslant \log x/\log\alpha)$, with the minor difference that the X_i are now on $\{A, \ldots, B - 1\}$. Thus with

$$Q(t_1, t_2, E) = (\alpha - 1)^E \sum_{\bar{v} \in V(t_1, t_2, E)} \prod_{A}^{B-1} (1/v_j!),$$

and with $V(t_1, t_2, E) = V(t_2, E) \setminus V(t_1, E)$, we have

$$\text{(9.6)} \quad Q(t_1, t_2, E) = (\alpha - 1)^E (B - A)^E (E!)^{-1} \mathrm{Prob}\left(\frac{\log t_1}{\log\alpha} < \sum_1^E X_i \leqslant \frac{\log t_2}{\log\alpha}\right).$$

Now Bernstein's theorem says that if Y_i, $1 \leqslant i \leqslant E$, are independent random variables uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$ then

$$\mathrm{Prob}\left(\left|\sum_1^E Y_i\right| \geqslant |b|\right) \leqslant 2\left(\rho\left(\frac{1}{2} - \frac{b}{E}\right)\right)^E.$$

Again we introduce further random variables $W_1, \ldots, W_E$ uniformly distributed on $[0, 1]$, and let $Y_i = (X_i + W_i - \frac{1}{2}(B + A))(B - A)^{-1}$. Then these Y_i are independent and uniformly distributed on $[-\frac{1}{2}, \frac{1}{2}]$. As in §8, for T and $V \geqslant 0$,

$$\operatorname{Prob}\left(T - V \leqslant \sum_1^E X_i \leqslant T\right) \leqslant \operatorname{Prob}\left((T - V - EA)(B - A)^{-1} - \frac{1}{2}E \right.$$
$$\left. \leqslant \sum_1^E Y_i \leqslant (T - EA + E)(B - A)^{-1} - \frac{1}{2}E\right).$$

For any real q_1, q_2 with $-\frac{1}{2}E \leqslant q_1 < q_2 \leqslant \frac{1}{2}E$ let $b(q_1, q_2) = q_1$ if $|q_1| < |q_2|$, q_2 otherwise as before, and

$$\operatorname{Prob}\left(\sum_1^E Y_i \in [q_1, q_2]\right) \leqslant 2(q_2 - q_1)\operatorname{Prob}\left(\left|\sum_1^E Y_i\right| \geqslant |b|\right).$$

Now

$$K(x, y) \leqslant x \sum_{E=1}^{2\log x} \sum_{j=1}^{[\log x/\log_2 x]} \sum_{V(x\log^{-j} x,\, x\log^{1-j} x,\, E)} \log^{1-j} x$$
$$\cdot \sum_{U(x)} \prod_1^C 2^{-ju_j} \binom{u_j + 2^j - 1}{u_j} \prod_A^{B-1} (1/v_j!).$$

From Lemma 9.2, then

$$K(x, y) \leqslant \left\{x \sum_{E=1}^{2\log x} \sum_{j=1}^{[\log x/\log_2 x]} \sum_{V(x\log^{-j} x,\, x\log^{1-j} x,\, E)} \log^{1-j} x \prod_A^{B-1} (1/v_j!)\right\}$$
$$\cdot \exp\left(O(\log_2 x)^2\right).$$

For the inmost sum we get an upper bound on its log of

$$\sqrt{\log x}\left(-h\log r - h\log h + h + h\log\rho\left(\frac{1}{2} - \frac{r}{h} + \frac{r_j\log_2 x}{h\log x}\right)\right)$$
$$-j\log_2 x + O\left((1 + r)^2\log_2 x\right) \tag{9.7}$$

as in §8.

Now let $g(h, r, s) = -h\log r - h\log h + h + h\log\rho(\frac{1}{2} - rs/h) + s\sqrt{\log x} - \sqrt{\log x}$ for $0 \leqslant s \leqslant 1$ and $h > rs$. Then for $r \leqslant \text{const}\cdot\log^{1/4} x$,

$$\sup_{0\leqslant s\leqslant 1,\, h > rs} g(h, r, s) = f(r). \tag{9.8}$$

PROOF. We have $g(h, r, s, x) = g(h, rs) + h\log s + (s - 1)\sqrt{\log x} \leqslant f(rs) + (s - 1)\sqrt{\log x}$. We claim that for r small enough, this is $\leqslant f(r)$.

Now $d(f(rs) + (s - 1)\sqrt{\log x})/ds = rf'(rs) + \sqrt{\log x} \geqslant rf'(r) + \sqrt{\log x}$ since f is concave and decreasing. Since from §7, $f'(r) = -(r^{-1}h(r) + \tau)$, $\tau(\frac{1}{2} - r/h(r)) \geqslant -(1 - r/h(r))^{-1}$ and $1 - r/h(r) \geqslant C_{17}r^{-1}$, $rf'(r) \geqslant -h(r) - r^2/C_{17}$. Since $h(r) = O(1 + r)$ from (7.5), there is $C_{19} > 0$ such that $rf'(r) + \sqrt{\log x} > 0$ for $r \leqslant C_{19}\log^{1/4} x$. This proves (9.8) since the claimed supremum is clearly attained with $s = 1$, $h = h(r)$.

Now in (9.7) the largest value possible is $f(r)\sqrt{\log x} + O((1+r)^2 \log_2 x)$ with $s = 1$, $j = 1$, and $h = h(r)$. There are fewer than $(\log x)^2$ summands in the upper bound for $K(x, y)$, so

$$K(x, y) \leqslant x \log^2 x \cdot \exp\left(f(r)\sqrt{\log x} + O\left((1+r)^2 \log\log x\right) + O(\log\log x)^2\right)$$

which is equivalent to Theorem 9.1. □

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